Risk contributions of lambda quantiles

A. INCE[*,ʻ], I. PERI[*,†] and S. PESENTI[‡]

[*]Department of Economics, Mathematics and Statistics, Birkbeck, University of London, Malet Street, Bloomsbury, London WC1E 7HX, UK
[‡]Department of Statistical Sciences, University of Toronto, 700 University Avenue, Toronto, ON M5G 1Z5, Canada

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Risk contributions of portfolios form an indispensable part of risk adjusted performance measurement. The risk contribution of a portfolio, e.g., in the Euler or Aumann-Shapley framework, is given by the partial derivatives of a risk measure applied to the portfolio profit and loss in direction of the asset units. For risk measures that are not positively homogeneous of degree 1, however, known capital allocation principles do not apply. We study the class of lambda quantile risk measures that includes the well-known Value-at-Risk as a special case but for which no known allocation rule is applicable. We prove differentiability and derive explicit formulae of the derivatives of lambda quantiles with respect to their portfolio composition, that is their risk contribution. For this purpose, we define lambda quantiles on the space of portfolio compositions and consider generic (also non-linear) portfolio operators.

We further derive the Euler decomposition of lambda quantiles for generic portfolios and show that lambda quantiles are homogeneous in the space of portfolio compositions, with a homogeneity degree that depends on the portfolio composition and the lambda function. This result is in stark contrast to the positive homogeneity properties of risk measures defined on the space of random variables which admit a constant homogeneity degree. We introduce a generalised version of Euler contributions and Euler allocation rule, which are compatible with risk measures of any homogeneity degree and non-linear but homogeneous portfolios. These concepts are illustrated by a non-linear portfolio using financial market data.

Keywords: Lambda Quantiles; Capital Allocation; Risk Contribution; Lambda Value-at-Risk; Euler Allocation

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ʻCorresponding author. Email: ainc02@student.bbk.ac.uk
†Email: i.peri@bbk.ac.uk
‡Email: silvana.pesenti@utoronto.ca

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