Aggregated Markov chain models in life insurance: properties and valuation

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Abstract

In multi-state life insurance, the random pattern of states of the insured $Z = \{Z(t)\}_{t\geq 0}$ is usually assumed to follow a time-inhomogeneous Markov jump process on a finite state space $\mathcal{J} = \{1, \ldots, J\}$ with intensity matrices $\boldsymbol{M}(t) = \{\mu_{ij}(t)\}_{i,j\in\mathcal{J}}$. This implies that the jump times in the associated marked point process $(T_n, Y_n)_{n\in\mathbb{N}}$ have the conditional distributions

$$\mathbb{P}(T_{n+1} > t \,|\, (T_i, Y_i)_{i\leq n}) = \exp\left(\int_{T_n}^{t} \mu_{Y_n Y_n}(x)\mathrm{d}x\right).$$

In the context of inhomogeneous phase-type distributions (IPH), cf. [1], the conditional sojourn time $T_{n+1} - T_n \mid (T_i, Y_i)_{i\leq n}$ is seen to have a one-dimensional IPH distribution, independent of past sojourn times and transitions.

In this talk, we introduce a so-called aggregated Markov chain model, where we assign a number $d_i \in \mathbb{N}$ of micro states to each biometric macro state $i \in \mathcal{J}$, leading to sojourn times in macro states being dependent and IPH distributed of general dimension. This extension is compactly obtained by extending the transition intensity matrices to have block-partitioned forms:

$$\boldsymbol{M}(s) = \begin{pmatrix} \boldsymbol{M}_{11}(s) & \boldsymbol{M}_{12}(s) & \cdots & \boldsymbol{M}_{1J}(s) \\ \boldsymbol{M}_{21}(s) & \boldsymbol{M}_{22}(s) & \cdots & \boldsymbol{M}_{2J}(s) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \boldsymbol{M}_{J1}(s) & \boldsymbol{M}_{J2}(s) & \cdots & \boldsymbol{M}_{JJ}(s) \end{pmatrix},$$

where $\boldsymbol{M}_{ii}(s)$ are sub–intensity matrices of dimension $d_i \times d_i$ describing transitions between micro states of macro state i, and $\boldsymbol{M}_{ij}(s)$, $j \neq i$, are non–negative matrices of dimension $d_i \times d_j$ describing transitions from macro state i to j.

Although this model in general leads to path dependencies in the states of the insured, we show that the properties of IPHs give explicit and tractable expressions for the distribution of jump times and transitions in terms of so-called product integrals of matrix functions, cf.

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e.g. [4]. These properties are then used to derive similar matrix expressions for expected life insurance cash flows and corresponding reserves in a setup where payments to the insured are allowed to depend on the duration in the different macro states, naturally extending the matrix representations obtained in [2].

From our general results, we identify an important special case where the block transition matrices are on the form

$$\boldsymbol{M}_{ij}(s) = \boldsymbol{\beta}_{ij}(s)\boldsymbol{\pi}_j(s),$$

where $\boldsymbol{\beta}_{ij}(s)$ are d_i-dimensional column vectors and $\boldsymbol{\pi}_j(s)$ are d_j-dimensional probability (row) vectors. This special case is shown to give a certain kind of reset property where sojourn time distributions are independent of past jump times and transitions. We show that this implies a specific semi-Markovian structure on the states of the insured, and we then link this special case to the classic semi-Markovian life insurance models known from e.g. [3, 5].

We end the talk by presenting a numerical example of the results, which serves to illustrate applicability of our results to examples known from existing literature and actuarial practice.

Keywords: Multi-state life insurance; Aggregated Markov chains; Inhomogeneous phase-type distributions; Product integrals; Semi-Markov processes.

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