Numerical Approximations of Gerber-Shiu Functions in a Markovian Shot-Noise Environment

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Abstract

One major simplification of the famous Cramér-Lundberg model is the assumption of a constant intensity of the underlying counting process. This issue can be resolved by replacing the Poisson process by a Cox process, a process with stochastic intensity. Our approach is to use a ruin model, whose counting process is driven by a Markovian shot-noise process. For this, we consider a Poisson process N^λ with constant intensity $\rho > 0$ and some positive *i.i.d.* random variables Y_i independent of N^λ. Then, the Markovian shot-noise process is given by

$$\lambda_t = \lambda_0 e^{-\gamma t} + \sum_{i=1}^{N_t^\lambda} Y_i e^{-\gamma(t-T_i)},$$

with some positive initial intensity λ_0 and positive decay parameter γ. Let N be a Cox process with intensity λ and U_i positive *i.i.d.* random variables independent of λ and N. Using these, we define the surplus process of the Markovian shot-noise model as

$$X_t = x + ct - \sum_{i=1}^{N_t} U_i,$$

where $x > 0$ is the starting capital and $c > 0$ the constant premium rate. The combined process (X, λ) is a piecewise-deterministic Markov process, whose generator is given by

$$\mathcal{A}f(x,\lambda) = \delta_\phi f(x,\lambda) + \lambda \int_0^\infty f(x-u,\lambda)\, F_U(\mathrm{d}u) + \rho \int_0^\infty f(x,\lambda+y)\, F_Y(\mathrm{d}y) - (\lambda+\rho) f(x,\lambda),$$

where $\delta_\phi f(x,\lambda) = \lim_{h\to 0} \frac{f(x+ch,\lambda e^{-\gamma h}) - f(x,\lambda)}{h}$ denotes the path derivative of the function f.

By the approach of Gerber and Shiu, the classical risk measure, namely the ruin probability, can be generalized to the family of Gerber-Shiu functions. Let $w(x, y)$ be a bounded and

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continuous function, $\tau = \inf\{t > 0 \mid X_t \leq 0\}$ the time of ruin and $\kappa \geq 0$. Then, the corresponding Gerber-Shiu function is defined by

$$g_\kappa(x,\lambda) = \mathbb{E}_{(x,\lambda)}\left[w(X_{\tau-}, -X_\tau) I_{\{\tau<\infty\}} e^{-\kappa\tau}\right].$$

A broadly applicable way to determine this, is to use Monte Carlo techniques to simulate the paths of the processes and calculate the corresponding values. Alternatively, one can exploit the fact that, in Markovian structures, expectations can often be rewritten as solutions of Feynman-Kac-type equations. In the case of our model, we show that the GS-functions are in the domain of the generator of the PDMP (X,λ), and that they are the unique bounded solutions of

$$\begin{aligned}\delta_\phi f(x,\lambda) + &\lambda \int_0^x f(x-u,\lambda)\, F_U(\mathrm{d}u) + \lambda \int_x^\infty w(x, u-x)\, F_U(\mathrm{d}u) \\ &+ \rho \int_0^\infty f(x,\lambda+y)\, F_Y(\mathrm{d}y) - (\lambda+\rho+\kappa) f(x,\lambda) = 0.\end{aligned}$$

Choosing a suitable grid, we approximate the above equations by the following system of linear equations,

$$\begin{aligned}0 = &\frac{f(x_{i+1},\lambda_{j-1}) - f(x_i,\lambda_j)}{h} + \lambda_j \sum_{k=1}^{\min(i-1,N_U)} f(x_{i-k},\lambda_j) p_k^U \\ &+ \rho \sum_{k=1}^{N_Y} f(x_i,\lambda_{j+k}) p_k^Y(j) - (\lambda_j+\rho+\kappa) f(x_i,\lambda_j).\end{aligned}$$

To show that these numerical approximations converge to the corresponding values of the GS-functions, we use the weak convergence of processes. For this, we identify the calculated values with GS-functions of certain approximating Markov chains and show that these converge in distribution against the original PDMP (X,λ). Since the GS-functions are continuous and bounded functionals of the corresponding processes, the numerical approximations converge against the desired values.

Keywords: Ruin Theory, Piecewise-Deterministic Markov Processes, Cox Processes

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